Is time continuous?

P. A. Varotsos[1,*]

[1]*Solid State Section and Solid Earth Physics Institute, Physics Department, University of Athens, Panepistimiopolis, Zografos 157 84, Athens, Greece*

Conventional time is modelled as the one dimensional continuum R^1 of real numbers. This continuity, however, does *not* stem from *any* fundamental principle. On the other hand, natural time is *not* continuous and its values as well as those of the energy, form *countable* sets, i.e., with cardinalities either finite or equal to $\aleph_0$, where this symbol stands for the *transfinite* number of natural numbers. For infinitely large number of events, the values of natural time form a *denumerable* set, i.e., its cardinality is exactly $\aleph_0$, while those of conventional time an *uncountable* set. This has a drastically larger cardinality, which in the light of the continuum hypothesis becomes equal to $2^{\aleph_0}$.

PACS numbers: 01.70.+w, 05.40.-a, 06.30.Ft, 02.10.Ab

In reviewing the state of physics today, a consensus seems to emerge that we are missing something absolutely fundamental, e.g., [1, 2]. Furthermore, there is a widespread belief that, it is not space but time that in the end poses the greatest challenge to science (e.g., p.18 of [3]). Time, according to Weyl (see p.5 of [4]) for example, is "the primitive form of the stream of consciousness. It is a fact, however, obscure and perplexing to our minds, that . . . one does not say this *is* but this is *now*, yet no more" or according to Gödel "that mysterious and seemingly self-contradictory being which, on the other hand, seems to form the basis of the world's and our own existence." (p.111 of [3]). The challenge seems to stem from the fact that special relativity and quantum mechanics, which are the two great (and successful) theories of twentieth-century physics, are based on entirely different ideas, which are not easy to reconcile (In general, the former theory, according to Einstein[5], is an example of "principled theory" in the sense that you start with the principles that underlie the theory and then work down to deduce the facts, while the latter is a "constructive theory" meaning that it describes phenomena based on some known facts but an underlying principle to explain the strangeness of the quantum world has not yet been found). In particular, special relativity puts space and time on the same footing, but quantum mechanics treats them very differently, e.g., see p.858 of Ref.[6]. (In quantum gravity, space is fluctuating and time is hard to define, e.g., [7]). More precisely, as far as the theory of special relativity is concerned, let us recall the following wording of Einstein[8]:

"Later, H. Minkowski found a particularly elegant and suggestive expression........, which reveals a formal relationship between Euclidean geometry of three dimensions and the space time *continuum* of physics........ From this it follows that, in respect to its *role* in the equations of physics, though not with regard to its physical significance, time is equivalent to the space co-ordinates (apart from the relations of reality). From this point of view, physics is, as it were, Euclidean geometry of four dimensions, or, more correctly, a static in a four-dimensional Euclidean *continuum*."

whereas in quantum mechanics, Von Neumann complains[9]:

"First of all we must admit that this objection points at an essential weakness which is, in fact, the *chief weakness* of quantum mechanics: its non-relativistic character, which distinguishes the time t from the three space coordinates x,y,z, and presupposes an objective simultaneity concept. In fact, while all other quantities (especially those x,y,z, closely connected with t by the Lorentz transformation) are represented by operators, there corresponds to the time an ordinary number-parameter t, just as in classical mechanics".

Note also that Pauli[10] has earlier shown that there is no operator canonically conjugate to the Hamiltonian, if the latter is bounded from below. This means that for many systems a time operator does *not* exist. In other words, the introduction of an operator t is basically forbidden and the time must necessarily be considered as an ordinary number (but recall the long standing question that Schrödinger's equation, as well as Einstein's general theory of relativity, is symmetric under time reversal in contrast to the fact that our world is *not*, e.g., [11]). These observations have led to a quite extensive literature mainly focused on time-energy (as well as on "phase-action") uncertainty relation, proposing a variety of attempts to overcome these obstacles. The discussion of this literature, however, lies beyond the scope of the present paper. We just summarize here that the (conventional) time t is *currently* modelled as the one-dimensional *continuum* R^1 of the real numbers, e.g., p. 10 of [7] (or p.12 of [4] in which it is stated that ". . . the straight line . . . is homogeneous and a linear *continuum* just like time"). It is this *continuity* on which the present paper is focused, in a sense that will be explained below.

It has been recently shown that novel dynamical features hidden behind time series in complex systems can emerge upon analyzing them in a new time domain, termed natural time[12, 13, 14, 15, 16, 17] (see also below). It seems that this analysis enables the study of the dynamical *evolution* of a complex system and identifies

*Electronic address: pvaro@otenet.gr

when the system enters a critical stage. Hence, natural time may play a key role in predicting impending catastrophic events in general. Relevant examples of data analysis in this new time domain have been presented in a large variety of fields including biology, earth sciences and physics. As a first example, we mention the analysis of the electrocardiograms which may herald a cardiac arrest[18, 19]. Secondly, the detection and the analysis of precursory electric signals, termed Seismic Electric Signals (e.g., [20, 21, 22, 23, 24]), may lead[12, 13, 16, 17] to the prediction of an impending strong earthquake. A third application of natural time refers to the manifestation of aging and scaling properties in seismic event correlation[25, 26]. Finally, as a fourth example we mention that the data of the avalanches of the penetration of magnetic flux into thin films of type II superconductors as well as those of a three dimensional pile of rice getting progressively closer to the critical state conform to[27] the features suggested, on the basis of natural time, to describe critical dynamics.

In a time series comprising N events, the *natural time* $\chi_k = k/N$ serves as an index for the occurrence of the k-th event[12, 13], and it is smaller than, or equal to, unity (cf. the symbol χ originates from the ancient Greek word $\chi\rho\acute{o}\nu o\varsigma$ (chronos) which means "time"). In natural time analysis the evolution of the pair of two quantities (χ_k, E_k) is considered, where E_k denotes in general a quantity proportional to the energy of the individual event. For example, to perform the analysis of seismic events (Fig.1(b)), we consider the time evolution of the pair (χ_k, M_{0_k}) where M_{0_k} stands for the seismic moment of the kth event[12, 16, 17], since M_{0_k} is proportional to the energy emitted in that earthquake (cf. M_{0_k} differs essentially from the magnitude M, but they are interconnected). As another example, we refer to the analysis of dichotomous electric signals (Fig.1(a)) where we consider E_k as being proportional to the duration of the kth pulse[13, 14, 15]. For the purpose of analysis, the following *continuous* function $\Phi(\omega)$ was introduced[12, 13]:

$$\Phi(\omega) = \frac{\sum_{k=1}^{N} E_k \exp\left(i\omega \frac{k}{N}\right)}{\sum_{n=1}^{N} E_n} = \sum_{k=1}^{N} p_k \exp\left(i\omega \frac{k}{N}\right) \quad (1)$$

where $p_k = E_k / \sum_{n=1}^{N} E_n$ and $\omega = 2\pi\phi$, and ϕ stands for the frequency in natural time, termed *natural* frequency. We then compute the power spectrum $\Pi(\omega)$ of $\Phi(\omega)$ as

$$\Pi(\omega) = |\Phi(\omega)|^2 \quad (2)$$

If we regard p_k as a probability density function, $\Phi(\omega)$ may be justified to be treated mathematically as a characteristic function in analogy with the probability theory[28]. Then, the properties of the distribution of p_k can be estimated by the expansion of this characteristic function for $\omega \longrightarrow 0$.

The optimality of the natural time representation of time series has been recently shown[29] by means of the following procedure: The structure of the time-frequency representation[30] of the signals was studied by employing the Wigner function[31] to compare the natural time representation with the ones, either in conventional time or other possible time reparametrizations. Significant enhancement of the signal was found in the time-frequency space if natural time is used, in marked contrast to other time domains. Since in time series analysis, it is desired to reduce uncertainty and extract signal information as much as possible, the most useful time domain should maximize the information measure, and hence minimize the entropy. This was statistically ascertained in natural time, by investigating a multitude of different time domains in several electric signals.

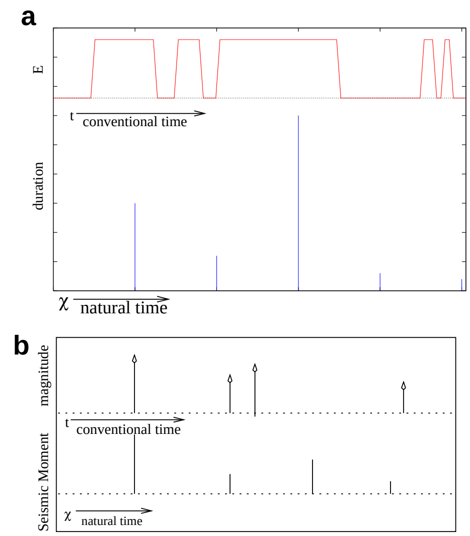

FIG. 1: (color online) (a) How a dichotomous series of electric pulses in conventional time t (upper panel, red) can be read[12, 13] in natural time χ (lower panel, blue). The symbol E stands for the electric field (*not* to be confused with E_k used in the text). (b) The same as in (a), but for a series of seismic events(e.g. [12, 16, 17]).

Natural time χ, from its definition, is *not continuous* and takes values which are *rational* numbers in the range (0,1]. (In these numbers, as the complex system evolves, the numerators are just the natural numbers (except 0), which denote the *order* of the appearance of the consecutive events). Hence, one of the fundamental differences between (conventional) time and natural time refers to the fact that the former is based on the idea of *continuum*, while the latter is *not*. This paper aims at raising some consequences of this difference, and in particular those that stem from the *set theory* developed by Cantor, having in mind the following remark made by Schrödinger (see pp. 62-63 of Ref.[32]):

"We are familiar with the idea of *continuum*, or we believe ourselves to be. We are *not* familiar with the enormous difficulty this concept presents to the mind, unless we have studied very modern mathematics (Dirichlet, Dedekind,Cantor)."

We clarify in advance that we do not tackle here the case (since it is inapplicable to our universe[33]) raised by Gödel in 1949 who discovered[34] unexpected solution to the equations of general relativity corresponding to universes in which no universal temporal ordering is possible(see also Refs.[3, 35] and references therein). This solution acquires its simplest form (see p.86 in [36]) "with *two* of the coordinate-line-elements time-like (the other two space-like)". Interestingly, Schrödinger in an early version of Ref.[36], which was published almost simultaneously with Gödel's work, had also emphasized that "there is no necessity for just three of the four line-elements being space-like, one time-like ..." (cf. note also that very recently it was suggested[37] that when the universe was small enough to be governed by quantum mechanics, it had four spatial dimensions and *no* dimension of time.)

We now recapitulate some points of the Cantor's set theory that are relevant to our present discussion. A *transfinite number or transfinite cardinal* is the cardinality of some *infinite* set, where the term *cardinality* of a set stands for the number of members it contains, e.g., pp. 2-3 of [38]. The set of natural numbers is labeled by N, while the number of natural numbers is designated by $\aleph_0$, i.e., $\aleph_0 = |N|$ (cf. the cardinality of a set S is labelled $|S|$). In this transfinite number, the zero subscript is justified by the fact that, as proved by Cantor, no infinite set has a smaller cardinality than the set of natural numbers. It can be shown that the set of rational numbers designated by Q has the same cardinality as the set of natural numbers, or $|N| = |Q|$ (e.g., Theorem 2 in [38]). In other words, the rationals are *exactly* as numerous as the naturals. Note that a set is *countable* iff its cardinality is either finite or equal to $\aleph_0$ and in particular is termed *denumerable* iff its cardinality is exactly $\aleph_0$. (cf. As usually, for "if and only if" we write simply "iff"). A set is *uncountable* iff its cardinality is greater than $\aleph_0$, see also below. Hence, natural time takes values (which, as mentioned, are rational numbers) that form in general a *countable* set; this becomes a *denumerable* set[39] if the number of events is infinitely large. Further, since in natural time analysis we consider the pairs (χ_k, E_k), the values of the quantity E_k *should* form a set with cardinality equal to (or smaller than) $\aleph_0$. In other words, the values of the energy also form a *countable* set, which reflects of course that the energy is *not continuous.*

The fact that $|N| = |Q|$ is an astounding result in view of the following: The rational numbers are *dense* in the real numbers, which means that between any two rational numbers on the real number line we can find *infinitely more* rational numbers. In other words, although the set of rational numbers seems to contain infinities within infinities, there are just as many natural numbers as there are rational numbers. This reflects the following point: Let us assume that we follow a system with some (experimental) accuracy, in which, as mentioned, after an infinite number of events the cardinality of the set of the values of natural time is $\aleph_0$. Let us assume that we now repeat the measurement with more sensitive instrumentation (i.e., counting events above an appreciably smaller energy or duration threshold in Figs. 1b and 1a, respectively) and hence between two consecutive events of the former measurement a considerable number of appreciably smaller events is monitored. The corresponding cardinality, in contrast to our intuition, is again $\aleph_0$. (The inverse, i.e., when the instrumentation becomes less sensitive, may correspond to a "coarse graining" procedure). In others words, when considering the occurrence of infinitely large number of consecutive events, the natural time takes values that form a *denumerable* set and this remains so even upon increasing the accuracy (and hence lowering the uncertainty) of our measurement.

We now turn to the aspects of Cantor's set theory related to the real numbers, which as mentioned are associated with the conventional time. It is shown that the number of points on a finite line segment is the same as the number of points on an infinite line (e.g., Theorem 13 in [38]). Considering the definition: The number of real numbers is the same as the number of points on an infinite line (or in the jargon, the *numerical continuum* has the same cardinality as the *linear continuum*), let "c" designate the cardinality of the continuum -or equivalently the cardinality of the set of real numbers. (Hence $c = |R|$ by definition). It is proven (e.g., Theorem 16 in [38]) that the set of real numbers is uncountable, or $|R| > \aleph_0$. (Equivalently, this theorem asserts that $c > \aleph_0$). Hence, the values of conventional time form an *uncountable* set, in contrast to that of natural time which as mentioned is countable. In order to further inspect this fundamental difference, we resort to the continuum hypothesis (CH) -see below- which was formulated (but not proved) by Cantor. This, after Euclid's parallel postulate, was the first major conjecture to be proved *undecidable* by standard mathematics[39].

We first clarify that the power set $^*S$ of a set S, which is the set of all subsets of S, has a cardinality $|^*S| = 2^{|S|}$ when S is finite. According to Cantor's Theorem the cardinality of the power set of an *arbitrary* set has a greater cardinality than the original arbitrary set, i.e., $|^*S| > |S|$ (e.g., Theorem 4 in [38]). This theorem is trivial for finite sets, but fundamental for infinite sets. Hence, for any infinite cardinality, there is a larger infinite cardinality, namely, the cardinality of its power set. From CH, which asserts that there is *no* cardinal number α such that $\aleph_0 < \alpha < c$, it follows that the next largest transfinite cardinal after $\aleph_0$ (labelled $\aleph_1$) is c, thus $c = \aleph_1$. Since Cantor proved (e.g., Theorem 17 in [38]) that $\aleph_1 = 2^{\aleph_0}$, CH leads to: $c = 2^{\aleph_0}$ (thus, this is the number of points on an infinite line). Hence, if we assume CH, the cardinality of the set of the values of natural time -when considering infinitely large number of events- corresponds to $\aleph_0$, while that of the conventional time is $2^{\aleph_0}$. The values of the former, as mentioned, are rational numbers, while almost all the values of the latter are *irrational*, because, since $2^{\aleph_0} \gg \aleph_0$, almost all reals are irrational numbers. (On the other hand, without assuming CH we have essentially no idea which transfinite number corresponds to c, and we would know the cardinality of the naturals, integers, and rationals, but *not* the cardinality of the reals, e.g., p. 15 of [38]). As for the values of E_k, they are *not* necessarily rational, because in general when taking $\aleph_0$ (at the most) out of $2^{\aleph_0}$ values they may all be irrational. Hence, even upon gradually improving the accuracy of our measurements, both sets $\{\chi_k\}$ and $\{E_k\}$ remain denumerable, the former consisting of rational numbers only.

Schrödinger, in order to point out the "Intricacy of the continuum", used an example analogous to the Cantor set C (see pp. 138-143 of Ref.[40]). The latter is given by taking the interval [0,1], removing the open middle third, removing the middle third of each of the two remaining pieces, and continuing this procedure ad infinitum. The cardinality of this set C is no less than that of [0,1]. Since C is a subset of [0,1], its cardinality is also no greater, so it must in fact be equal. In other words, there are as many points in the Cantor set as there are in [0,1], and the Cantor set is uncountable. The same holds for the *random* Cantor set, which has been suggested as being involved in the geometrical description of the fluctuations of the vacuum[41] (cf. modern physicists hypothesize that what appears to our senses as empty space is in reality a richly dynamical medium [42, 43], which has energy, e.g., [44, 45].). Hence the cardinality of either a Cantor set or a random Cantor set differs drastically from that of the set of the values of natural time.

We finally comment on the common view that (conventional) time is continuous, keeping in the frame that, as pointed out by Schrödinger (p. 145 of Ref.[46]) "our sense perceptions constitute our sole knowledge about things". In short, it seems that the continuity of time does not stem from *any* fundamental principle, but probably originates from the following general demand on continuity discussed by Schrödinger (see p. 130 of [40]):

"From our experiences on a large scale........ physicists had distilled the one clear-cut demand that a truly clear and complete description of any physical happening has to fulfill: it ought to inform you precisely of what happens at any point in space at any moment of time .... We may call this demand the *postulate of continuity of the description*". Schrödinger, however, subsequently commented on this demand as follows (see p.131 of [40]): "It is this postulate of continuity that appears to be unfulfillable!......" and furthermore added: "We *must not admit the possibility of continuous observation*". If we attempt a generalization of these intuitive remarks, we may say that the concept of natural time is not inconsistent with Schrödinger's point of view.

In summary, conventional time is currently assumed continuous, but this does not necessarily result from any fundamental principle. Its values form an uncountable set, almost all of which are *irrational* numbers. On the other hand, natural time is not continuous, and its values form a countable set consisting of rational numbers only; further, the values of the energy, which are not necessarily rational, also form a countable set. Upon considering an infinitely large number of events, the cardinality of the set of the values of natural time is $\aleph_0$ (cf. it persists even upon increasing the accuracy of the measurement), thus being drastically smaller than that of conventional time, which equals to $2^{\aleph_0}$ if we accept the validity of the continuum hypothesis.

---

[1] Concluding the 23^{rd} Solvey Conference (Dec. 2005), David Gross, compared the state of Physics today to that during the first Solvay conference in 1911 and said: "They were missing something absolutely fundamental" he said. "We are missing perhaps something as profound as they were back then", see New Scientist, December 10, pp.6,7.

[2] G. 't Hooft, in *Does God play dice*? Physics World, December 2005, (http://physicsweb.org/articles/world).

[3] P. Yourgrau, *A World Without Time:The forgotten legacy of Goedel and Einstein* (Basic Books, Cambridge MA, 2005).

[4] H. Weyl, *Space-Time-Matter* (Dover, New York, 1952).

[5] A. Einstein, *Ideas and Opinions* (Crown Publishers, 1954; new edition by Souvenir Press, 2005); see also p.54 in A. Gefter in New Scientist, 10 December, 2005.

[6] F. Wilczek, Rev. Mod. Phys. **77**, 857 (2005).

[7] F. Wilczek, PHYSICS TODAY **58**, 10, July (2005).

[8] A. Einstein, NATURE (London) **106**, 782 (1921).

[9] J. von Neumann, *Mathematical foundations of Quantum Mechanics* (Princeton University Press, Princeton N.J., 1955).

[10] W. Pauli, *Die allgemeinen Prinzipien der Wellenmechanik.* in K. Geiger and H. Scheel (eds), p.140 Handbuck der Physik 2nd Edition, Vol. 245 (Berlin Springer, 1933).

[11] O. Penrose, NATURE (London) **438**, 919 (2005).

[12] P. A. Varotsos, N. V. Sarlis, and E. S. Skordas, Practica of Athens Academy **76**, 294 (2001).

[13] P. A. Varotsos, N. V. Sarlis, and E. S. Skordas, Phys. Rev. E **66**, 011902 (2002).

[14] P. A. Varotsos, N. V. Sarlis, and E. S. Skordas, Phys. Rev. E **67**, 021109 (2003).

[15] P. A. Varotsos, N. V. Sarlis, and E. S. Skordas, Phys. Rev. E **68**, 031106 (2003).

[16] P. A. Varotsos, N. V. Sarlis, H. K. Tanaka, and E. S. Skordas, Phys. Rev. E **72**, 041103 (2005).

[17] P. A. Varotsos, N. V. Sarlis, E. S. Skordas, H. K. Tanaka, and M. S. Lazaridou, Phys. Rev. E **73**, 031114 (2006).

[18] P. A. Varotsos, N. V. Sarlis, E. S. Skordas, and M. S. Lazaridou, Phys. Rev. E **70**, 011106 (2004).

[19] P. A. Varotsos, N. V. Sarlis, E. S. Skordas, and M. S. Lazaridou, Phys. Rev. E **71**, 011110 (2005).

[20] P. A. Varotsos, K. Alexopoulos, K. Nomikos, and M. S. Lazaridou, NATURE (London) **322**, 120 (1986).

[21] P. Varotsos and K. Alexopoulos, *Thermodynamics of Point Defects and their Relation with Bulk Properties* (North Holland, Amsterdam, 1986).

[22] P. A. Varotsos, N. V. Sarlis, and E. S. Skordas, Phys. Rev. Lett. **91**, 148501 (2003).
[23] P. Varotsos, *The Physics of Seismic Electric Signals* (TerraPub, Tokyo, 2005).
[24] P. A. Varotsos, N. V. Sarlis, and E. S. Skordas, Appl. Phys. Lett **86**, 194101 (2005).
[25] U. Tirnakli and S. Abe, Phys. Rev. E **70**, 056120 (2004).
[26] S. Abe, U. Tirnakli, and P. A. Varotsos, Europhysics News **36/6**, 206 (2005).
[27] N. V. Sarlis, P. A. Varotsos, and E. S. Skordas, Phys. Rev. B **73**, 054504 (2006).
[28] W. Feller, *An Introduction to Probability Theory and Its Applications, Vol. II* (Wiley, New York, 1971).
[29] S. Abe, N. V. Sarlis, E. S. Skordas, H. Tanaka, and P. A. Varotsos, Phys. Rev. Lett. **94**, 170601 (2005).
[30] L. Cohen, *Time-Frequency Analysis: Theory and Applications* (Prentice-Hall, Upper Saddle River NJ, 1994).
[31] E. Wigner, Phys. Rev. **40**, 749 (1932).
[32] E. Schrödinger, *Nature and the Greeks* (Cambridge Univ. Press, Cambridge 1954; Canto Edition with *Science and Humanism*, 1996).
[33] K. Hentschel, PHYSICS TODAY **58**, 60, December (2005).
[34] K. Gödel, Rev. Mod. Phys. **21**, 447 (1949).
[35] M. Davis, NATURE (London) **435**, 19 (2005).
[36] E. Schrödinger, *Space-Time Structure* (Cambridge Univ. Press, Cambridge, 1985).
[37] S. W. Hawking and T. Hertog, arXiv:hep-th/0602091; see also A. Gefter, New Scientist, p.28, April 22, 2006.
[38] P. Suber, `http://www.earlham.edu/~peters/writing/infapp.htm`, published in the *St. John's Review*, **XLIV**, 2, 35-59 (1998).
[39] See EPAPS Document No. [...] for additional information. This document may be retrieved via the EPAPS homepage (`http://www.aip.org/pubservs/epaps.html`) or from `ftp.aip.org` in the directory /epaps/. See the EPAPS homepage for more information.
[40] E. Schrödinger, *Science and Humanism* (Cambridge Univ. Press, Cambridge 1951; Canto Edition with *Nature and the Greeks*, 1996).
[41] M. S. El Naschie, Chaos, Solitons and Fractals **19**, 209 (2004).
[42] F. Wilczek, NATURE (London) **433**, 239 (2005).
[43] F. Wilczek, PHYSICS TODAY **59**, 10, May (2006).
[44] A. Vilenkin, Published online 4 May 2006; 10.1126/Science.1128570.
[45] M. Carrol, NATURE (London) **440**, 1132 (2006).
[46] E. Schrödinger, *Mind and Matter* (Cambridge Univ. Press, Cambridge 1958; Canto Edition with *Autobiographical Sketches* and *What is Life*, Tenth printing, 2003).