Supplementary Information for the paper "Some properties of the entropy in the natural time"

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This supplementary information provides, in more detail, the mathematical proofs concerning the positivity, concavity and uniform continuity (or as usually called Lesche Stability) for both the variance κ_1 and the entropy S in the natural time. It also gives some additional comments on points discussed in the main text.

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In section I, we present some background material, while sections II, III and IV provide the proof, in detail, of the positivity, concavity and Lesche stability, respectively, of both the variance κ_1 and the entropy S in the natural time domain. Section V is reserved for the presentation of a more general theorem, while in Table I we give the date and the station at which the SES activities and AN, depicted in Fig.1 of the main text, have been recorded.

I. BACKGROUND MATERIAL

We first review some of the basic properties of the real functions $g(x) = x^2$ and $f(x) = x \ln x$ defined on the closed interval [0,1] (more accurately we consider $f(x) = \{x \ln x \, \forall \, x \in (0,1], 0 \text{ if } x = 0\}$). These are depicted in Fig.1, and note that the following two inequalities hold:

$$0 \leq g(x) \leq 1, \tag{1}$$

$$0 \geq f(x) \geq -\frac{1}{e}. \tag{2}$$

We now proceed to two very simple Lemmas:

Lemma 1 *Both $g(x)$ and $f(x)$ are continuous in the interval [0,1].*

Proof: For $g(x)$ this is trivial; for $f(x)$ it is also trivial for $x \in (0,1]$ and since $\lim_{x\to 0} f(x) = 0$, $f(x)$ is also continuous at $x = 0$.

Lemma 2 *Both $g(x)$ and $f(x)$ are convex in the interval (0,1].*

Proof: It is suffiecient to show that the second derivatives of these twice differentiable functions are positive. Indeed $g''(x) = 2$ and $f''(x) = 1/x$ which are both positive for $x > 0$.

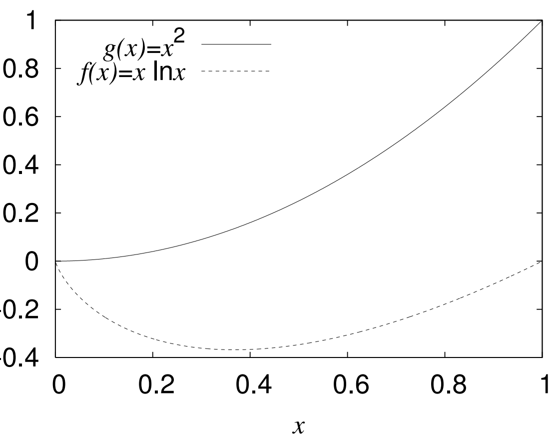

FIG. 1: The functions $g(x)$ and $f(x)$ in the closed interval [0,1].

TABLE I: The date and station at which the SES activities and AN, depicted in Fig.1 of the main text, have been recorded. For the location of the stations see Ref.[1]

Signal	Station	Date	Time(UT)
T1	MYT	04 Apr. 2003	14:10
C1	MYT	02 Apr. 2003	12:25
P1	ROD	26 Oct. 2003	03:34
P2	ROD	04 Nov. 2003	19:39
E1	KER	05 Oct. 2003	04:33
n7	LAM	12 Jan. 2003	08:14
n8	LAM	22 Aug. 2002	16:37
n9	LAM	25 Aug. 2002	09:19
n10	LAM	17 Aug. 2002	16:48
n11	LAM	12 Oct. 2003	07:40
n12	IOA	10 Jan. 2004	18:58
n13	PIR	09 Mar. 2004	04:44
n14	PIR	12 Mar. 2004	07:33

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II. POSITIVITY OF κ_1 AND S

We recall that

$$\kappa_1 = \langle \chi^2 \rangle - \langle \chi \rangle^2, \tag{3}$$

$$S = \langle \chi \ln \chi \rangle - \langle \chi \rangle \ln \langle \chi \rangle, \tag{4}$$

where the symbol $\langle\ \rangle$ stands for

$$\langle F(\chi) \rangle = \sum_{k=1}^{N} p_k F(\frac{k}{N}) \tag{5}$$

and p_k denotes:

$$p_k = \frac{Q_k}{\sum_{n=1}^{N} Q_n}. \tag{6}$$

In order to prove the positivity of κ_1 and S, we shall make use of the following well known theorem[2] (see also **12.411** at page 1101 of Ref. [3]):

Theorem 1 *(Jensen's inequality) If F is a convex function on the interval $[a, b]$, then*

$$F\left(\sum_{k=1}^{n} \lambda_k x_k\right) \leq \sum_{k=1}^{n} \lambda_k F(x_k)$$

where $0 \leq \lambda_k \leq 1$, $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$ and each $x_k \in [a, b]$.

Due to Lemma 2 both $g(x) = x^2$ and $f(x) = x \ln x$ are convex in [0,1]. Using in Jensen's inequality $\lambda_k = p_k$, $x_k = k/N$ and Eq.(5), we obtain :

$$\langle \chi \rangle^2 \leq \langle \chi^2 \rangle \tag{7}$$

and

$$\langle \chi \rangle \ln \langle \chi \rangle \leq \langle \chi \ln \chi \rangle, \tag{8}$$

respectively.

Obviously, Eqs. (7) and (8) imply the positivity of both κ_1 and S. Another important property of κ_1 and S is that they are not only bounded from below by zero, but also bounded from above by N-independent bounds:

$$0 \leq \kappa_1 = \langle \chi^2 \rangle - \langle \chi \rangle^2 \leq \langle \chi^2 \rangle + \langle \chi \rangle^2 \leq \sum_{k=1}^{N} p_k \left(\frac{k}{N}\right)^2 + 1 < 2 \tag{9}$$

due to Eq.(1),

$$0 \leq S = \langle \chi \ln \chi \rangle - \langle \chi \rangle \ln \langle \chi \rangle \leq |\langle \chi \ln \chi \rangle| + |\langle \chi \rangle \ln \langle \chi \rangle| \leq \sum_{k=1}^{N} p_k |\frac{k}{N} \ln \frac{k}{N}| + \frac{1}{e} < \frac{2}{e}, \tag{10}$$

due to Eq.(2).

III. THE CONCAVITY OF κ_1 AND S

The concavity of κ_1 and S with respect to p_k is straighforward[4] since they both have negative second derivatives:

$$\frac{\partial^2 \kappa_1}{\partial p_k \partial p_l} = -\frac{k\, l}{N^2}, \tag{11}$$

$$\frac{\partial^2 S}{\partial p_k \partial p_l} = -\frac{k\, l}{N^2} \left(\sum_{m=1}^{N} p_m \frac{m}{N}\right)^{-1}. \tag{12}$$

IV. LECHE STABILITY OF κ_1 AND S

Lesche stability[5] is considered[6–9] as an important property to be satisfied by an entropic measure $\Sigma[p]$. Following Ref.[6], Lesche stability implies that for two slightly different distributions $\{p_i\}_{i=1,2,\ldots N}$ and $\{p'_i\}_{i=1,2,\ldots N}$, the corresponding entropic measures $\Sigma[p]$ and $\Sigma[p']$ do not change drastically (and also in a uniform way, see below). Mathematically

$$\forall \epsilon > 0\ \exists \delta\ : \|p - p'\| < \delta \Rightarrow \left|\frac{\Sigma p - \Sigma p'}{\Sigma_{max}}\right| < \epsilon \tag{13}$$

for any value of N, with the metric $\|p\| = \sum_{i=1}^{N} |p_i|$ and Σ_{max} is the maximum value of Σ.

We note[7] that, for a fixed value of N, Lesche Stability implies uniform continuity which is a rather trivial statement, because a continuous function on a compact set is automatically uniformly continuous (Heine 1870, see below). It was pointed out[9] that Lesche condition is a definition of natural uniform metric continuity. The power of Lesche stability condition arises from the fact that uniform continuity may not survive in the $N \to \infty$ limit[8]. Thus, to avoid confusion, one should consider[8] that the mapping $\Sigma[p]$, where $p \in (\mathcal{R}^+)^N$, taken as a function of N, converges to a uniformly continuous function in a uniform manner, i.e., $\forall \epsilon > 0$ there exists δ_ϵ (which depends *only* on ϵ) such that $\forall p, p' \in (\mathcal{R}^+)^N$ and for every $N \in \mathcal{Z}^+$

$$\|p - p'\| < \delta_\epsilon \Rightarrow \left| \frac{\Sigma[p] - \Sigma[p']}{\Sigma_{max}} \right| < \epsilon. \tag{14}$$

In our case of κ_1 and S, there is at least one distribution $\{p_i\}_{i=1,2,...N}$, the constant one with all $p_i = 1/N$, for which for all N the corresponding values $\kappa_{1,c}$ and S_c,:

$$\kappa_{1,c}(N) = \sum_{k=1}^{N} \frac{k^2}{N^3} - \left(\sum_{k=1}^{N} \frac{k}{N^2} \right)^2,$$

$$S_c(N) = \sum_{k=1}^{N} \frac{k}{N^2} \ln\left(\frac{k}{N}\right) - \sum_{k=1}^{N} \frac{k}{N^2} \ln\left(\sum_{l=1}^{N} \frac{l}{N^2} \right),$$

as well as, in the limit $N \to \infty$:

$$\lim_{N\to\infty} \kappa_{1,c}(N) = \kappa_{1,u} = \frac{1}{12}, \lim_{N\to\infty} S_c(N) = S_u = \frac{\ln 2}{2} - \frac{1}{4}.$$

obtain well defined finite and positive values. We note that both $\kappa_{1,c}(N)$ and $S_c(N)$ are monotonically increasing with respect to N and hence:

$$\frac{1}{16} = \kappa_{1,c}(2) \le \kappa_{1,c}(N), \ \frac{5\ln 2 - 3\ln 3}{4} = S_c(2) \le S_c(N).$$

Since Σ_{max} should be by definition greater or equal than each of these values for all N, we can replace Σ_{max} in the definition of Lesche stability by either $\frac{1}{16}$ or $\frac{5\ln 2 - 3\ln 3}{4}$, respectively. Then, these positive numbers can be absorbed in ϵ and thus we retain the usual definition of uniform metric continuity in a uniform manner (independent of N). This is what we shall prove:

$$\forall \epsilon > 0, N \in Z^+ \exists \delta(\epsilon) : \|p - p'\| < \delta(\epsilon) \Rightarrow |\Sigma[p] - \Sigma[p']| < \epsilon. \tag{15}$$

Proposition 1 *(Stability of κ_1) The variance κ_1 in the natural time:*

$$\kappa_1[p] = \sum_{k=1}^{N} p_k \left(\frac{k}{N}\right)^2 - \left(\sum_{k=1}^{N} \frac{k}{N} p_k \right)^2 \tag{16}$$

satisfies the condition (15), and hence is Lesche stable.

Proof: For every $\epsilon > 0$, we can consider $\delta(\epsilon) = \epsilon/3$ so that if $\|p - p'\| < \delta(\epsilon)$ we have:

$$\begin{aligned}
|\kappa_1[p] - \kappa_1[p']| &= \left| \sum_{k=1}^{N} \left(\frac{k}{N}\right)^2 (p_k - p'_k) - \left(\sum_{k=1}^{N} \frac{k}{N} p_k \right)^2 + \left(\sum_{k=1}^{N} \frac{k}{N} p'_k \right)^2 \right| = \\
&= \left| \sum_{k=1}^{N} \left(\frac{k}{N}\right)^2 (p_k - p'_k) + \left(\sum_{k=1}^{N} \frac{k}{N} p_k \right) \sum_{k=1}^{N} \frac{k}{N}(p'_k - p_k) + \left(\sum_{k=1}^{N} \frac{k}{N} p'_k \right) \sum_{k=1}^{N} \frac{k}{N}(p'_k - p_k) \right| \le \\
&\le \left| \sum_{k=1}^{N} \left(\frac{k}{N}\right)^2 (p_k - p'_k) \right| + \left| \sum_{k=1}^{N} \frac{k}{N} p_k \right| \left| \sum_{k=1}^{N} \frac{k}{N}(p'_k - p_k) \right| + \left| \sum_{k=1}^{N} \frac{k}{N} p'_k \right| \left| \sum_{k=1}^{N} \frac{k}{N}(p'_k - p_k) \right| \le \\
&\le \sum_{k=1}^{N} \left(\frac{k}{N}\right)^2 |p_k - p'_k| + \left| \sum_{k=1}^{N} \frac{k}{N}(p'_k - p_k) \right| + \left| \sum_{k=1}^{N} \frac{k}{N}(p'_k - p_k) \right| \le \\
&\le \sum_{k=1}^{N} |p_k - p'_k| + \sum_{k=1}^{N} \left|\frac{k}{N}\right| |p'_k - p_k| + \sum_{k=1}^{N} \left|\frac{k}{N}\right| |p'_k - p_k| \le \\
&\le 3 \sum_{k=1}^{N} |p_k - p'_k|
\end{aligned} \tag{17}$$

but since $\|p - p'\| = \sum_{k=1}^{N} |p_k - p'_k| < \epsilon/3$, inequality (17) implies that

$$|\kappa_1[p] - \kappa_1[p']| < \epsilon \tag{18}$$

which completes the proof.

Now, before proceeding to the final proof for the stability of the entropy S, we make use of a well known theorem[10]:

Theorem 2 *(Heine 1870) If a function $F(x)$ of a real variable x is continuous when $a \le x \le b$, then $F(x)$ is uniformly continuous throughout the range $a \le x \le b$.*

In Lemma 1 we proved that $f(x) = \{x \ln x \,\forall\, x \in (0,1], 0 \text{ if } x = 0\}$ is continuous in the closed interval [0,1], and hence it is also uniformly continuous in the same interval. Uniform continuity implies that

$$\forall \frac{\epsilon}{2} > 0, x, y \in [0,1]\, \exists \delta_1(\epsilon/2) \,:\, |x - y| < \delta_1(\epsilon/2) \Rightarrow |x \ln x - y \ln y| < \frac{\epsilon}{2}. \tag{19}$$

Now, we can show that S is Lesche stable.

Proposition 2 *(Stability of S) The entropy S in the natural time:*

$$S[p] = \sum_{k=1}^{N} p_k \frac{k}{N} \ln \frac{k}{N} - \left(\sum_{k=1}^{N} p_k \frac{k}{N} \right) \ln \sum_{k=1}^{N} p_k \frac{k}{N} \tag{20}$$

satisfies the condition (15), and hence it is Lesche stable.

Proof: For every $\epsilon > 0$, we can consider $\delta(\epsilon) = \min\left[\frac{e\epsilon}{2}, \delta_1(\epsilon/2)\right]$ so that if $\|p - p'\| < \delta(\epsilon)$ we have:

$$\begin{aligned} |S[p] - S[p']| &= \left| \sum_{k=1}^{N} (p_k - p'_k) \frac{k}{N} \ln \frac{k}{N} - \left(\sum_{k=1}^{N} \frac{k}{N} p_k \right) \ln \sum_{k=1}^{N} \frac{k}{N} p_k + \left(\sum_{k=1}^{N} \frac{k}{N} p'_k \right) \ln \sum_{k=1}^{N} \frac{k}{N} p'_k \right| \leq \\ &\leq \left| \sum_{k=1}^{N} (p_k - p'_k) \frac{k}{N} \ln \frac{k}{N} \right| + |x \ln x - y \ln y| \,, \end{aligned} \tag{21}$$

where $x = \sum_{k=1}^{N} \frac{k}{N} p_k$ and $y = \sum_{k=1}^{N} \frac{k}{N} p'_k$. We now consider that

$$|x - y| = \left| \sum_{k=1}^{N} \frac{k}{N} (p_k - p'_k) \right| \leq \sum_{k=1}^{N} \left| \frac{k}{N} \right| |p_k - p'_k| \leq \sum_{k=1}^{N} |p_k - p'_k| < \delta(\epsilon) \leq \delta_1(\epsilon/2) \tag{22}$$

and hence (see condition (19))

$$|x \ln x - y \ln y| < \frac{\epsilon}{2}. \tag{23}$$

Now, we return to inequality (21) to complete the proof:

$$\begin{aligned} |S[p] - S[p']| &\leq \left| \sum_{k=1}^{N} (p_k - p'_k) \frac{k}{N} \ln \frac{k}{N} \right| + |x \ln x - y \ln y| < \\ &< \left| \sum_{k=1}^{N} (p_k - p'_k) \frac{k}{N} \ln \frac{k}{N} \right| + \frac{\epsilon}{2} \leq \\ &\leq \sum_{k=1}^{N} |p_k - p'_k| \left| \frac{k}{N} \ln \frac{k}{N} \right| + \frac{\epsilon}{2} \leq \\ &\leq \sum_{k=1}^{N} |p_k - p'_k| \frac{1}{e} + \frac{\epsilon}{2}, \end{aligned} \tag{24}$$

since we assumed $\|p - p'\| = \sum_{k=1}^{N} |p_k - p'_k| < \delta(\epsilon) \leq \frac{e\epsilon}{2}$, the inequality (24) becomes:

$$|S[p] - S[p']| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{25}$$

which means that the condition (15) is obeyed for S, i.e., S is Lesche stable.

V. A MORE GENERAL THEOREM

The following general theorem seems to hold.

Theorem 3 *Let $F(x) : [0,1] \to \mathcal{R}$ which is:*

1. *uniformly continuous in [0,1]*
2. *strictly convex in (0,1]*
3. *twice differentiable in (0,1]*

then the functional:

$$\Sigma[p] = \sum_{k=1}^{N} p_k F\left(\frac{k}{N}\right) - F\left(\sum_{k=1}^{N} p_k \frac{k}{N}\right)$$

is:

1. *positive*
2. *concave*
3. *Lesche stable.*

Proof:

Positivity: Since $F(x)$ is a convex function in the interval (0,1], we apply Jensen's inequality (**Theorem 1**) with $\lambda_k = p_k$ and $x_k = k/N$ and directly obtain:

$$F\left(\sum_{k=1}^{N} p_k \frac{k}{N}\right) \leq \sum_{k=1}^{N} p_k F\left(\frac{k}{N}\right) \Rightarrow \Sigma[p] \geq 0$$

Concavity: For $\Sigma[p]$ we have

$$\frac{\partial \Sigma[p]}{\partial p_k} = F\left(\frac{k}{N}\right) - \frac{k}{N} F'\left(\sum_{k=1}^{N} p_k \frac{k}{N}\right),$$

where $F'(x)$ is the first derivative of $F(x)$, and

$$\frac{\partial^2 \Sigma[p]}{\partial p_l \partial p_k} = -\frac{l}{N}\frac{k}{N} F''\left(\sum_{k=1}^{N} p_k \frac{k}{N}\right). \qquad (26)$$

Since $F(x)$ is convex and twice differentiable, then(e.g. see **12.41** in page 1100 of Ref.[3])) its second derivative is positive $F''(x) \geq 0$. Thus, Eq.(26) implies[4] that $\frac{\partial^2 \Sigma[p]}{\partial p_l \partial p_k}$ is negative and hence $\Sigma[p]$ is concave.

Lesche Stability: Following Ref.[5, 8], we shall prove that $\forall \epsilon > 0$ there exists δ_ϵ (which depends *only* on ϵ) such that $\forall\, p, p' \in (\mathcal{R}^+)^N$ and for every $N \in \mathcal{Z}^+$

$$\|p - p'\| < \delta_\epsilon \Rightarrow \left| \frac{\Sigma[p] - \Sigma[p']}{\Sigma_{max}} \right| < \epsilon. \qquad (27)$$

Let us first discuss about Σ_{max}. By defining $\left\{p_i = c_i \equiv \frac{1}{N}\right\}_{i=1,2,\ldots N}$, we have

$$\Sigma_c(N) = \sum_{k=1}^{N} \frac{1}{N} F\left(\frac{k}{N}\right) - F\left(\sum_{k=1}^{N} \frac{k}{N^2}\right)$$

which as N tends to infinity is strictly positive, because $F(x)$ is strictly convex, and moreover finite since:

$$|\Sigma_c(N)| = \left| \sum_{k=1}^{N} \frac{1}{N} F\left(\frac{k}{N}\right) - F\left(\sum_{k=1}^{N} \frac{k}{N^2}\right) \right| \leq \sum_{k=1}^{N} \frac{1}{N} \left| F\left(\frac{k}{N}\right) \right| + \left| F\left(\sum_{k=1}^{N} \frac{k}{N^2}\right) \right| \leq M + M = 2M,$$

where M is an upper bound for $F(x)$, which always exists(since $F(x)$ is uniformly continuous in [0,1]).

Moreover, for the same reason

$$\lim_{N\to\infty} \Sigma_c(N) = \Sigma_u = \int_0^1 F(x)dx - F\left(\frac{1}{2}\right),$$

where $\Sigma_u > 0$, due the fact that $F(x)$ is strictly convex. Thus, $\Sigma_c(N)$ considered as a real sequence has the property $\Sigma_c(N) \neq 0$ (strictly convex) and $\lim_{N\to\infty} \Sigma_c(N) = \Sigma_u \neq 0$, then (see **4-21** in page 61 of Ref.[11]):

$$I = \inf\left\{\Sigma_c(N) \;:\; N \in \mathcal{N}\right\} > 0, \qquad (28)$$

and thus

$$\Sigma_{max} \geq \Sigma_c(N) \Rightarrow \frac{1}{\Sigma_c(N)} \geq \frac{1}{\Sigma_{max}},$$

but

$$I < \Sigma_c(N) \Rightarrow \frac{1}{I} > \frac{1}{\Sigma_c(N)},$$

and hence

$$\frac{1}{I} > \frac{1}{\Sigma_{max}}, \qquad (29)$$

where I is a well defined positive real number, the infimum of the positive sequence $\Sigma_c(N)$.

Moreover, since $F(x)$ is uniformly continuous, we have:

$$\forall \frac{\epsilon I}{2} > 0, x, y \in [0,1]\; \exists \delta_2(\epsilon I/2) \;:\; |x - y| < \delta_2(\epsilon I/2) \Rightarrow |F(x) - F(y)| < \frac{\epsilon I}{2}. \qquad (30)$$

.

We now proceed to the proof of the Lesche stabily, see condition (27): For every $\epsilon > 0$, we can condider $\delta(\epsilon) = \min\left[\frac{\epsilon I}{2M}, \delta_2(\epsilon I/2)\right]$ so that if $\|p - p'\| < \delta(\epsilon)$ we have:

$$|\frac{\Sigma[p]-\Sigma[p']}{\Sigma_{max}}| < \frac{1}{I}\left|\sum_{k=1}^{N}(p_k - p'_k)F\left(\frac{k}{N}\right) - F\left(\sum_{k=1}^{N}\frac{k}{N}p_k\right) + F\left(\sum_{k=1}^{N}\frac{k}{N}p'_k\right)\right| \leq$$
$$\leq \frac{\left|\sum_{k=1}^{N}(p_k - p'_k)F\left(\frac{k}{N}\right)\right| + |F(x) - F(y)|}{I}, \quad (31)$$

where $x = \sum_{k=1}^{N}\frac{k}{N}p_k$ and $y = \sum_{k=1}^{N}\frac{k}{N}p'_k$. We consider that

$$|x - y| = \left|\sum_{k=1}^{N}\frac{k}{N}(p_k - p'_k)\right| \leq \sum_{k=1}^{N}\left|\frac{k}{N}\right||p_k - p'_k| \leq \sum_{k=1}^{N}|p_k - p'_k| < \delta(\epsilon) \leq \delta_2(\epsilon I/2) \quad (32)$$

and hence (see condition (30))

$$|F(x) - F(y)| < \frac{\epsilon I}{2}.$$

We now return to inequality (31):

$$|\frac{\Sigma[p]-\Sigma[p']}{\Sigma_{max}}| < \frac{\left|\sum_{k=1}^{N}(p_k - p'_k)F\left(\frac{k}{N}\right)\right| + |F(x) - F(y)|}{I} <$$
$$< \frac{1}{I}\sum_{k=1}^{N}|p_k - p'_k||F\left(\frac{k}{N}\right)| + \frac{\epsilon}{2} \leq$$
$$\leq \frac{1}{I}\sum_{k=1}^{N}|p_k - p'_k|M + \frac{\epsilon}{2} <$$
$$< \frac{M}{I}\delta(\epsilon) + \frac{\epsilon}{2} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad (33)$$

which completes the proof.

[1] P. Varotsos, *The Physics of Seismic Electric Signals* (TERRAPUB, Tokyo, in press).

[2] For example, see `http://planetmath.org/encyclopedia/JensensInequality.html`.

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[4] All the matrix elements of the Hessian ($H_{k,l} = \partial^2/\partial p_k \partial p_l \Sigma[p]$) have the form $H_{k,l} = -\epsilon V_k V_l$, where $V = (1/N, 2/N, ..., 1)$ and $\epsilon > 0$. Such a Hessian cannot have a positive eigenvalue λ, because $H_{k,l}e_\lambda = \lambda e_\lambda \Rightarrow e_\lambda^T H_{k,l} e_\lambda = \lambda \|e_\lambda\| = \lambda = -\epsilon e_\lambda^T V^T V e_\lambda = -\epsilon \|V e_\lambda\| \leq 0$, where $e_\lambda (\in \mathcal{R}^N)$ is any normalized eigenvector of the symmetric real matrix $H_{k,l}$.

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