Some properties of the entropy in the natural time

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We show that the entropy S, defined as $S \equiv \langle \chi \ln \chi \rangle - \langle \chi \rangle \ln \langle \chi \rangle$ [Phys. Rev. E **68**, 031106 (2003)] where χ stands for the natural time [Phys. Rev. E **66**, 011902 (2002)], exhibits positivity and concavity as well as stability or experimental robustness. Furthermore, the distinction between the seismic electric signal activities and "artificial" noises, based on the classification of their S values, is lost when studying the time-reversed signals. This reveals the profound importance of considering the (true) time arrow.

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Among the many generalizations of the well known Boltzmann-Gibbs-Shannon (BGS) entropy $S_{\mathrm{BGS}} = -k\Sigma_{i=1}^{N} p_i \ln p_i$, one finds the Renyi entropy [1], the Tsallis entropy [2], the Abe entropy [3], the Landsberg-Vedral entropy [4], the Kaniadakis entropy [5,6], and the escort (or normalized Tsallis) entropy [7]. Much attention has been focused recently on the Tsallis entropy, which is currently considered as a milestone of the so-called nonextensive statistical mechanics. An entropic functional $\Sigma[p]$, where $\{p_i\}_{i=1,2,\ldots,N}$ is a probability distribution for a given system, should be *concave* (e.g., pp. 52–53 of Ref. [8]). For the central importance of this concavity on both the 0th and the 2nd principle of thermodynamics; see, for example, Ref. [9]. Thus, an investigation of the *concavity* of the aforementioned entropies has been made and the results are as follows: BGS, Tsallis, Abe, and Kaniadakis entropy are concave (e.g., Ref. [10] and references therein), the Renyi entropy and the Landsberg-Vedral entropy are concave only for $0<q<1$, while the escort entropy [11] is concave only for $q>1$, where q stands for the so called entropic index, i.e., the exponent q in the Tsallis entropy $S_q[p] = (\Sigma_{i=1}^{N} p_i^q - 1)/(1-q)$. Another important issue that has recently attracted strong interest is the *stability* or experimental robustness of these entropies, e.g., see Refs. [9,10,12–16]. In particular, this investigation is usually made in terms of an early suggestion by Lesche [17] (Lesche stability criterion), which states that an entropic measure is stable if its change upon an arbitrarily small deformation of the distribution (representing fluctuations of experimental data [12]) remains small. By means of this stability criterion, Lesche [17] showed that the BGS entropy is stable, while the Renyi entropy is unstable. Abe later proved [13] that the Tsallis entropy is also stable, while the escort entropy is not. Finally, the stability was also shown for the Kaniadakis entropy [12,14], while it became clear [11] that the Landsberg-Vedral entropy does not obey this criterion. To sum up, the BGS, Tsallis, Abe, and Kaniadakis entropies (beyond *positivity*, e.g., see Ref. [10]) exhibit concavity and are Lesche stable.

Recently, the entropy S in natural time has been suggested, which is defined as [18,19]

$$S \equiv \langle \chi \ln \chi \rangle - \langle \chi \rangle \ln \langle \chi \rangle, \tag{1}$$

where χ stands for the natural time. The natural time is introduced [19,20] by ascribing to the kth pulse of an electric signal consisting of N pulses the value $\chi_k = k/N$ and the analysis is made in terms of the couple (χ_k, Q_k), where Q_k stands for the duration of the kth pulse. In Ref. [18], one can find examples of how electric signals (consisting of pulses of dichotomous nature) are read in the natural time, i.e., p_k or $p(\chi)$ versus χ_k or χ, respectively, where $p_k = Q_k/\Sigma_{n=1}^{N} Q_n$. These examples include seismic electric signal (SES) activities, which are recorded well before major earthquakes, and "artificial" noise (AN), which is emitted from nearby artificial electrical sources. Excerpts of more recent examples, i.e., collected during the last few years, are depicted in Fig. 1. Some properties of the entropy S have been already presented in the Appendix of Ref. [21]. It is one of the aims (an

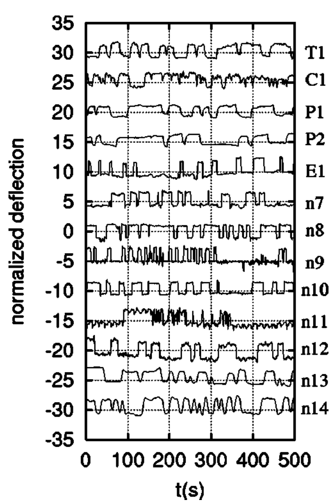

FIG. 1. Electric signals recorded (sampling rate $f_{\mathrm{expt}} = 1$ Hz) during the last few years: Excerpts of five SES activities labeled T1, C1, P1, P2, E1 and eight sets of artificial noise labeled n7 to n14. The electric field **E** is usually measured in mV/km, but here we present these signals in normalized units, i.e., by substracting the mean value μ and dividing by the standard deviation σ. All signals for the sake of clarity are displaced vertically by constant factors.

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additional one will be mentioned below) of the present paper to investigate whether the entropy S exhibits the previously mentioned important properties for the entropic measures, i.e., positivity, concavity, and stability (in the sense of Lesche's criterion). Aspects supporting the view that, in time-series analysis, the use of natural time—compared to other time domains—reduces uncertainty and extracts the maximum information possible, despite their importance, are beyond the scope of the present work and will be developed elsewhere.

SES activities and AN may look similar, but it has been found [18] that the entropy S can distinguish them as follows: If S_u (≈ 0.0966) denotes the entropy of a uniform distribution (as it was defined in Refs. [18,19,22]), the S values of the SES activities are smaller than S_u, while those of the AN are larger than (or equal to) S_u. If the corresponding S values do not markedly differ from S_u, the distinction should be better made by means of the complexity measures introduced in Ref. [23] that quantify the change of the fluctuations δS at different length scales.

It is currently believed (see Ref. [24] and references therein) that, in general, there is a relation between the irreversibility of *thermodynamic* processes as expressed by the breaking of time-reversal symmetry, and the entropy production in such processes. An essential characteristic of these processes is that the time-reversal invariance of the microscopic dynamics is apparently broken [24]. It means that out of equilibrium a particular sequence of macrostates and its time reversal can have very different plausibility (this, basically, must be the reason for the positivity of entropy production [24]). This, since both SES activities and AN are out of equilibrium processes, motivated us to investigate the following point: Is the aforementioned S criterion (i.e., $S<S_u$ for SES, while $S \geq S_u$ for AN) for the distinction between SES activities and AN still applicable, upon calculating the S values after a time reversal of the original time series? The answer to this question constitutes an additional aim of the present paper. We find that under time reversal S is not invariant and the aforementioned S criterion is not valid. We note that, in general, the proposal of the use of fractional time derivatives for subdiffussive transport also touches upon fundamental principles such as locality, irreversibility, and invariance under time translation [25] because fractional derivatives are nonlocal operators that are not invariant under time reversal. These issues, which are generally avoided [25] in relevant proposals based upon purely mathematical or heuristic aspects, were discussed in the context of long time limits and coarse graining [26]. It was then found([26]; see also [25] and references therein) that fractional derivatives with orders between 0 and 1 may appear, in general, as infinitesimal generators of a coarse grained macroscopic time evolution.

We first prove the positivity of S. Since the function $f(x)=\{x \ln x \, \forall x \in (0,1], 0 \text{ if } x=0\}$ is convex (see p. 92 of Ref. [27]), we consider Jensen's inequality [28] (see also Sec. 12.411 of Ref. [29]), which states that if $F(x)$ is a convex function on the interval $[a,b]$, then

$$F\left(\sum_{k=1}^{n} \lambda_k x_k\right) \leq \sum_{k=1}^{n} \lambda_k F(x_k), \tag{2}$$

where $0 \leq \lambda_k \leq 1$, $\lambda_1+\lambda_2+\cdots+\lambda_n=1$ and each $x_k \in [a,b]$. Using $\lambda_k=p_k$, $x_k=\chi_k(\equiv k/N)$ and $F(x)=f(x)$, Jensen's inequality yields

$$\langle\chi\rangle \ln\langle\chi\rangle \leq \langle\chi \ln \chi\rangle, \tag{3}$$

and hence

$$S \equiv \langle\chi \ln \chi\rangle - \langle\chi\rangle \ln\langle\chi\rangle \geq 0. \tag{4}$$

We now turn to the proof of the concavity of S with respect to p_k. Using the properties of the average natural time [19,21] $\langle\chi\rangle=\Sigma_{m=1}^{N}(m/N)p_m$ a direct differentiation of the equation

$$S \equiv \sum_{k=1}^{N} \frac{k}{N} p_k \ln\left(\frac{k}{N\sum_{l=1}^{N}(l/N)p_l}\right) \tag{5}$$

(which results from a combination of Eqs. (A12)–(A15) of Ref. [21]) with respect to p_k and p_l leads to

$$\frac{\partial^2 S}{\partial p_k \partial p_l} = -\frac{lk}{N^2\langle\chi\rangle}. \tag{6}$$

Since $\langle\chi\rangle$ is always positive, we find that the right side of Eq. (6) is always negative. This shows [30] the concavity of the entropy S with respect to p_k.

Following Ref. [12], Lesche stability implies, as mentioned, that for two slightly different distributions $\{p_i\}_{i=1,2,\ldots,N}$ and $\{p'_i\}_{i=1,2,\ldots,N}$, the corresponding entropic measures $\Sigma[p]$ and $\Sigma[p']$ do not change drastically. One should consider [31] that $\Sigma[p]$, where $p \in (\mathcal{R}^+)^N$, taken as a function of N, converges to a uniformly continuous function in a uniform manner, i.e., $\forall\ \epsilon>0$ there exists δ_ϵ (which depends *only* on ϵ) such that $\forall p, p' \in (\mathcal{R}^+)^N$ and for every $N \in \mathcal{Z}^+$

$$\|p-p'\| < \delta_\epsilon \Rightarrow \left|\frac{\Sigma[p]-\Sigma[p']}{\Sigma_{\max}}\right| < \epsilon, \tag{7}$$

with the metric $\|p\|=\Sigma_{i=1}^{N}|p_i|$ and $\Sigma_{\max}$ the maximum value of Σ.

In our case of S, there is at least one distribution $\{p_i\}_{i=1,2,\ldots,N}$, the constant one [21] $p_i=1/N$, for which, for all N, the corresponding entropy S_c is given by

$$S_c(N) = \sum_{k=1}^{N} \frac{k}{N^2} \ln\left(\frac{k}{N}\right) - \sum_{k=1}^{N} \frac{k}{N^2} \ln\left(\sum_{l=1}^{N} \frac{l}{N^2}\right), \tag{8}$$

which reaches a well defined finite and positive value in the limit $\lim_{N\to\infty} S_c(N)=S_u=(\ln 2)/2-1/4 \approx 0.0966$. By virtue of the fact that $S_c(N)$ is monotonically increasing with respect to N, we have that $\zeta \equiv (5 \ln 2 - 3 \ln 3)/4 = S_c(2) \leq S_c(N)$. Since $\Sigma_{\max} \geq S_c(N) \geq \zeta \Rightarrow 1/\zeta \geq 1/\Sigma_{\max}$, we can replace $\Sigma_{\max}$ in the relation (7) by ζ.

We now consider that the function $f(x)=\{x \ln x \, \forall x \in (0,1], 0 \text{ if } x=0\}$ is a continuous function defined on the compact set [0,1], and hence it is uniformly continuous. This reflects that $f(x)$ is also bounded, and one can see [31] that $|f(x)| \leqslant 1/e$. Moreover, uniform continuity implies that there exists $\delta_1(\epsilon) > 0$ so that for every $x, y \in [0,1]$

$$|x-y| < \delta_1(\epsilon) \Rightarrow |x \ln x - y \ln y| < \frac{\epsilon\zeta}{2}. \tag{9}$$

Now the proof of the Lesche stability of S proceeds as follows. Indeed, for every $\epsilon > 0$, we can consider $\delta(\epsilon) = \min[e\epsilon/2, \delta_1(\epsilon)]$ so that if $\|p-p'\| < \delta(\epsilon)$ we have

$$\left|\frac{\Sigma[p]-\Sigma[p']}{\Sigma_{\max}}\right| \leqslant \frac{1}{\zeta}\left|\sum_{k=1}^{N}(p_k-p'_k)\frac{k}{N}\ln\frac{k}{N} - x\ln x + y\ln y\right| \leqslant \left|\sum_{k=1}^{N}\frac{(p_k-p'_k)}{\zeta}\frac{k}{N}\ln\frac{k}{N}\right| + \frac{|x\ln x - y\ln y|}{\zeta}, \tag{10}$$

where $x = \sum_{k=1}^{N}(k/N)p_k$ and $y = \sum_{k=1}^{N}(k/N)p'_k$. We now take into account that

$$|x-y| = \left|\sum_{k=1}^{N}\frac{k}{N}(p_k-p'_k)\right| \leqslant \sum_{k=1}^{N}\left|\frac{k}{N}\right||p_k-p'_k| \leqslant \sum_{k=1}^{N}|p_k-p'_k| < \delta(\epsilon) \leqslant \delta_1(\epsilon) \tag{11}$$

and therefore [see condition (9)]

$$\frac{|x\ln x - y\ln y|}{\zeta} < \frac{\epsilon}{2}, \tag{12}$$

the consideration of which turns inequality (10) to (for more details see [31])

$$\left|\frac{\Sigma[p]-\Sigma[p']}{\Sigma_{\max}}\right| < \frac{1}{\zeta}\sum_{k=1}^{N}|p_k-p'_k|\left|\frac{k}{N}\ln\frac{k}{N}\right| + \frac{\epsilon}{2} \leqslant \frac{\sum_{k=1}^{N}|p_k-p'_k|}{\zeta e} + \frac{\epsilon}{2} < \frac{\delta(\epsilon)}{\zeta e} + \frac{\epsilon}{2} \leqslant \epsilon, \tag{13}$$

which completes the proof.

We now investigate the S value deduced upon analyzing in the natural time domain the time series of SES activities and AN obtained upon considering the time reversal $\mathcal{T}$ of the original time series, $\mathcal{T}p_k = p_{N-k+1}$. This, for the sake of convenience, will be designated by S_- (in contrast to the S value which results from the analysis of the original time series). Table I presents the S and S_- values of the SES activities and AN investigated in Ref. [18] as well as of the signals depicted in Fig. 1 (the date as well as the station at which each of the latter signals has been recorded can be found in Ref. [31]). For the sake of completeness, we also give in Table I the value of the variance $\kappa_1 \equiv \langle\chi^2\rangle - \langle\chi\rangle^2$ obtained in each case (note that κ_1 does not change upon time reversal—since it results from a power spectrum [20]—in a similar fashion as the exponents obtained from detrended fluctuation analysis [32,33] (DFA), and Hurst analysis [34]; see also below). An inspection of this table reveals the following. Although the S values are classified as stated above, i.e., $S < S_u$ for the SES activities and $S \gtrapprox S_u$ for AN, this does not hold, in general, for the S_- values. This is so, since for all the SES activities (with the probable exception of K2) we find that the S_- values are smaller than (or equal to) S_u, but for AN no *common* behaviour could be found, because S_- is either smaller or larger than S_u. In other words, no distinction between SES activities and AN can be achieved on the basis of S_- values alone. This means the following, if we recall that the S value takes into account the sequential order of pulses and hence captures elements of the dynamics hidden in this order [21,23]. Only when considering the (true) time arrow (i.e., analyzing in the natural time domain the time series as it was actually recorded in nature) can the S value pinpoint the difference in the dynamics between these two groups of electric signals. Recall that the SES activities are characterized by *critical* dynamics and hence exhibit infinitely ranged long range correlations, while in AN the intensity of the long-range correlations is markedly weaker [18]. Numerical studies of *critical* models do show that both S and S_- are smaller than S_u and will be published elsewhere.

TABLE I. The values of S, κ_1, S_- for the SES activities and AN analyzed in Ref. [18] as well as those depicted in Fig. 1.

Signal	S	κ_1	S_-
K1	0.067 ± 0.003[a]	0.063 ± 0.003[a]	0.074 ± 0.003
K2	0.081 ± 0.003[a]	0.078 ± 0.004[a]	0.103 ± 0.003
A	0.070 ± 0.008[a]	0.068 ± 0.004[a]	0.084 ± 0.008
U	0.092 ± 0.004[a]	0.071 ± 0.004[a]	0.071 ± 0.004
T1	0.088 ± 0.007	0.084 ± 0.007	0.098 ± 0.010
C1	0.083 ± 0.004	0.074 ± 0.002	0.080 ± 0.004
P1	0.087 ± 0.004	0.075 ± 0.004	0.081 ± 0.004
P2	0.088 ± 0.003	0.071 ± 0.005	0.072 ± 0.015
E1	0.087 ± 0.007	0.077 ± 0.017	0.081 ± 0.007
n1	0.143 ± 0.003[a]	0.115 ± 0.003[a]	0.127 ± 0.004
n2	0.103 ± 0.003[a]	0.093 ± 0.003[a]	0.122 ± 0.003
n3	0.117 ± 0.010[a]	0.100 ± 0.008[a]	0.118 ± 0.010
n4	0.106 ± 0.010[a]	0.100 ± 0.013[a]	0.138 ± 0.010
n5	0.091 ± 0.011[a]	0.086 ± 0.007[a]	0.120 ± 0.011
n6	0.102 ± 0.007[a]	0.084 ± 0.004[a]	0.095 ± 0.007
n7	0.116 ± 0.005	0.085 ± 0.005	0.083 ± 0.005
n8	0.117 ± 0.004	0.095 ± 0.007	0.099 ± 0.005
n9	0.110 ± 0.010	0.091 ± 0.005	0.095 ± 0.010
n10	0.112 ± 0.005	0.087 ± 0.007	0.087 ± 0.006
n11	0.122 ± 0.012	0.088 ± 0.007	0.079 ± 0.012
n12	0.104 ± 0.005	0.094 ± 0.005	0.103 ± 0.009
n13	0.124 ± 0.007	0.084 ± 0.007	0.077 ± 0.008
n14	0.124 ± 0.005	0.087 ± 0.005	0.081 ± 0.007

[a]From Ref. [18].

We now comment on the aforemenionted point that, in contrast to S, the variance κ_1 as well as the generalized variance σ_{DFA} of the DFA technique or the R/S function (Hurst analysis) do not change upon time reversal. In other words, only S does satisfy the condition to be "causal" in the following sense. When studying a dynamical system evolving in time [35], the "causality" of an operator describing this evolution assures that the values assumed by the operator, at each time instant, depend solely on the past values of the system. This reflects that a "causal" operator is able to represent the evolution of the system according to the (true) time arrow. The "causality" of an operator has the following two consequences. First, the operator can represent a real physical system evolving in time. Second, the operator can reveal the differences arising upon time reveral of the series. In order to overcome this "lack of causality" of DFA, an alternative approach termed the detrending moving average (DMA) has been suggested [36], motivated by an earlier study [37]. The DMA has been defined in terms of a generalized variance, analogously to the DFA, with the important difference of being able to be operated "causally" (i.e., in real time) and continuously (since the time series does not need an artifical subdivision into boxes). The extent to which the DMA, as well as a more recent approach [38]—motivated by the DMA—that is based on the analysis of clusters formed by the moving average of a long-range correlated time series, may reveal the differences arising in the time series of the SES activities and AN due to time reveral is currently under detailed investigation.

In summary, the entropy S does exhibit positivity, concavity, and Lesche stability. Interestingly, it also shows a breaking of the time-reversal symmetry and can classify similar looking electric signals of different dynamics *only* when analyzing their original (and not the time-reversed) time series.

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[31] See EPAPS Document No. E-PLEEE8-71-081503 for additional information. A direct link to this document may be found in the online article's HTML reference section. The document may also be reached via the EPAPS homepage (http://www.aip.org/pubservs/epaps.html) or from ftp.aip.org in the directory /epaps/. See the EPAPS homepage for more information.
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